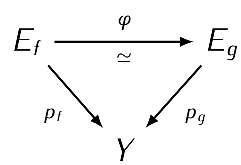
Fall 2024 MTH 727 Homework 3

1. Recall that if $f\colon X \to Y$ is a map then we can replace it by a Hurewicz fibration $p_f\colon E_f \to Y$ where

$$E_f = \{(x, \omega) \in X \times PY \mid f(x) = \omega(0)\} \subseteq X \times PY$$

and $p_f(x, \omega) = \omega(1)$. Let $f, g\colon X \to Y$ be maps such that $f \simeq g$. Show that there exists a homotopy equivalence $\varphi\colon E_f \to E_g$ such that the following diagram commutes:



2. Given maps $i\colon A \to X$ and $f\colon A \to Y$ define $X \cup_f Y = X \sqcup Y/\sim$ where $i(a) \sim f(a)$ for all $a \in A$. Show that if i is a cofibration then so is the the map $j\colon Y \to X \cup_f Y$.

3. Let $i\colon A \to X$ be a cofibration of compact Hausdorff spaces. Show that for any space Y the induced map

$$i^*\colon \mathrm{Map}(X, Y) \to \mathrm{Map}(A, Y)$$

is a Hurewicz fibration.

4. Consider the 3-dimensional torus $T = S^1 \times S^1 \times S^1$ as CW complex with a single 3-dimensional cell, and let $T^{(2)}$ be its 2-skeleton. The quotient space $T/T^{(2)}$ is then homeomorphic to the sphere S^3, so we can regard the quotient map $T \to T/T^{(2)}$ as a map $f\colon T \to S^3$. Since the induced homomorphism of homology groups $f_*\colon H_3(T) \to H_3(S^3)$ is an isomorphism, the map f is not homotopic to the constant map. Consider the composition $g = pf\colon T \to S^2$ where $p\colon S^3 \to S^2$ is the Hopf fibration. Show that the map g_* induces trivial homomorphisms on all homotopy groups, homology groups and cohomology groups, but it is not homotopic to the constant map. (Hint: show that if there was a homotopy $g \simeq *$ they we would also have a homotopy $f \simeq *$)

5. Prove Proposition 11.3.