# 18 | Spectral Sequences

**18.1 Motivation.** Hurewicz Isomorphism Theorem 17.4 lets us compute the first non-trivial homotopy group of a space $X$ using homological methods. In order to extend this to to higher homotopy groups of $X$, one can attempt the following approach. Assume that $\pi_k(X) = 0$ for $k < n$ and that we know homology groups of $X$. This in particular gives us $\pi_n(X) \cong H_n(X)$. We can construct a map $f_n \colon X \to K(\pi_n(X), n)$ which induces an isomorphism on $n$-th homotopy groups. Let $X_n$ denote the homotopy fiber of $f_n$. The long exact sequence of a fibration shows that $\pi_k(X_n) = 0$ for $k < n+1$ and $\pi_k(X) \cong \pi_k(X_n)$ for $k \geq n+1$. In particular using the Hurewicz Isomorphism Theorem we obtain

$$\pi_{n+1}(X) \cong \pi_{n+1}(X_n) \cong H_{n+1}(X_n)$$

Thus computations of $\pi_{n+1}(X)$ are reduced to computing a homology group of the space $X_n$.

This procedure can be repeated: once we know $\pi_{n+1}(X_n)$, we can construct a map $f_{n+1} \colon X_n \to K(\pi_{n+1}(X_n), n+1)$ that induces an isomorphism on $(n+1)$-st homotopy groups. Taking $X_{n+1}$ to be the homotopy fiber of this map we obtain isomorphisms

$$\pi_{n+2}(X) \cong \pi_{n+2}(X_n) \cong \pi_{n+2}(X_{n+1}) \cong H_{n+2}(X_{n+1})$$

Proceeding recursively, we obtain that in order to compute homotopy groups of $X$ it suffices to compute homology groups of spaces $X_k$ for $k \geq n-1$ such that $X_{n-1} = X$ and which are connected by fibration sequences

$$X_{k+1} \to X_k \to K(\pi_{k+1}(X), k+1)$$

In order to carry out this program we would need to;

- calculate homology groups of Eilenberg-MacLane spaces $K(G, k)$;
- given a fibration sequence $F \to E \to B$ find a relationship between homology groups of the spaces $F$, $E$, and $B$.

Spectral sequences provide a tool for achieving the second of these objectives. They are helpful with the first one as well.

In this chapter we give the definition of a spectral sequence and some examples how spectral sequences are used. Explanation in which circumstances spectral sequences occur is left for later.

**18.2 Definition.** A *bigraded* abelian group $G_{**}$ is a collection of abelian groups $G_{p,q}$ for $p, q \in \mathbb{Z}$.

**18.3 Definition.** A *(first quadrant, homological) spectral sequence* $(E^r_{**}, d^r)$ is a sequence of bigraded abelian groups $E^r_{**}$ for $r = 1, 2, \dots$ such that:

1) $E^r_{p,q} = 0$ if $p < 0$ or $q < 0$.
2) Each $E^r_{**}$ is equipped with homomorphisms (*differentials*)

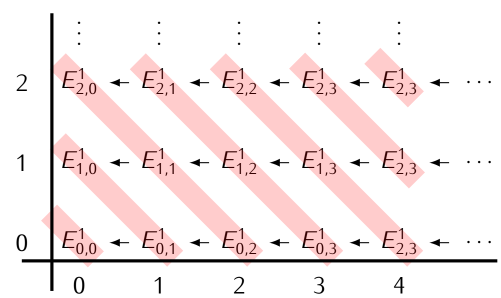
$$d^r \colon E^r_{p,q} \to E^r_{p-r,q+r-1}$$

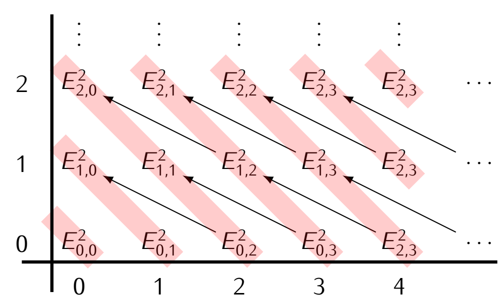
   satisfying $d^r d^r = 0$.
3) For each $r \geq 0$ we have $E^{r+1}_{p,q} \cong H_{p,q}(E^r_{**})$ where

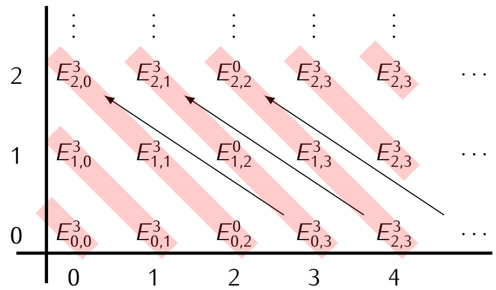
$$H_{p,q}(E^r_{**}) = \frac{\mathrm{Ker}(d^r \colon E^r_{p,q} \to E^r_{p-r,q+r-1})}{\mathrm{Im}(d^r \colon E^r_{p+r,q-r+1} \to E^r_{p,q})}$$

**18.4 Note.** The bigraded group $E^r_{**}$ is called the *r-th page* of the spectral sequence.

Below are pictures of the first three pages of a spectral sequence. Notice that the differentials $d^r$ always go between groups $E^r_{p,q}$ where $p + q = n$ for some $n$ and groups where $p + q = n - 1$.






Since all groups $E^r_{p,q}$ with negative $p$ or $q$ are trivial, the differentials $d^r$ originating at $E^r_{p,q}$ are trivial for $r > p$. Likewise, the differentials $d^r$ terminating at $E^r_{p,q}$ are trivial if $r > q+1$. As a consequence, for $r \leq \max(p+1, q+2)$ we get

$$E^r_{p,q} = E^{r+1}_{p,q} = E^{r+2}_{p,q} = \dots$$

For each $p, q$, let $E^\infty_{p,q}$ denote this recurring group. These groups form a bigraded group $E^\infty_{**}$.

In typical applications of spectral sequences, $E^\infty_{**}$ is related to some object of interest, e.g. homology groups of some space. This is done as follows. We start with a graded abelian group $H_*$ i.e. a collection of abelian groups $H_n$ for $n \in \mathbb{Z}$. A *filtration* of $H_*$ is a sequence of graded subgrops:

$$0 = F_{-1}H_* \subseteq F_0 H_* \subseteq F_1 H_* \subseteq \dots \subseteq H_*$$

such that $\bigcup_{p=0}^\infty F_p H_* = H_*$.

**18.5 Definition.** We say that a spectral sequence $(E^r_{**}, d^r)$ *converges* to a graded group $H_*$ if there exists a filtration of $H_*$ such that

$$E^\infty_{p,q} \cong F_p H_{p+q} / F_{p-1} H_{p+q}$$

for all $p, q$.

Results on existence spectral sequences usually say that there exists a spectral sequence for which we can say describe in some useful way groups $E^r_{p,q}$ for some fixed $r$, and that this sequence converges to some interesting graded group $H_*$. Here is one example of such a statement:

**18.6 Theorem.** *Let $p\colon E \to B$ be a Serre fibration and let $F = p^{-1}(b_0)$ for some $b_0 \in B$. If the space $B$ is simply connected then there exists a spectral sequence $(E^r_{**}, d^r)$ such that*

$$E^2_{p,q} \cong H_p(B, H_q(F))$$

*for all $p, q$, and which converges to $H_*(E)$.*

The spectral sequence described in this theorem is called the *Serre spectral sequence* of the fibration $p$.

The next result provides an example how spectral sequences are used in computations.

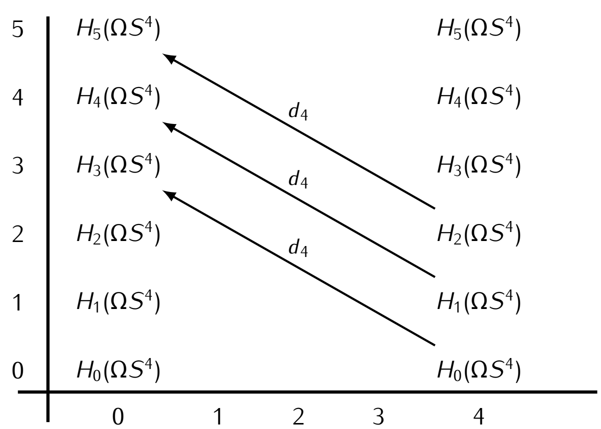
**18.7 Theorem.** *If $n \geq 2$ then*

$$H_m(\Omega S^n) \cong \begin{cases} \mathbb{Z} & \text{if } (n-1)|m \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* The space $\Omega S^n$ is the fiber of a Serre fibration $p\colon P \to S^n$ with a contractible space $P$. Consider the Serre spectral sequence of this fibration. We have

$$E^2_{p,q} \cong H_p(S^n, H_q(\Omega S^n)) \cong \begin{cases} H_q(\Omega S^n) & \text{if } p = 0, n \\ 0 & \text{otherwise} \end{cases}$$

For example, for $n = 4$ the second page of this spectral sequence looks as follows:




All differentials in the spectral sequence are trivial, except, possibly $d^n\colon E^n_{p,q} \to E^n_{0,q+n-1}$. It follows that $E^2_{**} = E^n_{**}$ and $E^{n+1}_{**} = E^\infty_{**}$. The total space $P$ of the fibration is contractible, so $H_0(P) = \mathbb{Z}$ and $H_p(P) = 0$ for $p > 0$. By Theorem 18.6 we have $E^\infty_{p,q} \cong F_pH_{p+q}(P)/F_{p-1}H_{p+q}(P)$ for some filtration $\{F_pH_*(P)\}$ of $H_*(P)$. It follows that

$$E^{n+1}_{p,q} = E^\infty_{p,q} \cong \begin{cases} \mathbb{Z} & \text{if } (p,q) = (0,0) \\ 0 & \text{otherwise} \end{cases}$$

Since $E^{n+1}_{p,q} \cong H_{p,q}(E^2_{**})$ we obtain that $H_0(\Omega S^n) \cong \mathbb{Z}$ and $H_p(\Omega S^n) = 0$ for $0 < p \leq n-2$. Also, all differentials $d^n$ must be isomorphisms. This gives:

$$H_p(\Omega S^n) \cong H_{p+(n-1)}(\Omega S^n) \cong H_{p+2(n-1)}(\Omega S^n) \cong H_{p+3(n-1)}(\Omega S^n) \cong \ldots$$

Taking $p = 0$ we obtain that $H_m(\Omega S^n) \cong \mathbb{Z}$ if $(n-1)|m$. In all other cases $H_m(\Omega S^n) \cong H_p(\Omega S^n)$ for some $0 < p \leq n-2$, and so it is a trivial group. □

**18.8 Note.** The proof of Theorem 18.7 used the observation that all differentials $d^r$ in the Serre spectral sequence of the fibration $p\colon P \to S^n$ were trivial for $r \geq n+1$. A situation like this appears frequently in computations involving spectral sequences, which motivates the next definition.

**18.9 Definition.** We say that a spectral sequence *collapses* at the page $r_0$ if all differentials $d^r$ are trivial for $r \geq r_0$.

If a spectral sequence collapses at the page $r_0$ then we have $E^{r_0}_{p,q} = E^{r_0+1}_{p,q} = \ldots = E^{\infty}_{p,q}$.