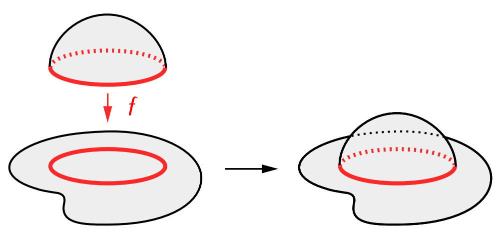
2 | Review: CW Complexes

2.1 Definition. Let X be a space and let $f\colon S^{n-1} \to X$ be a continuous function. We say that a space Y is obtained by *attaching an n-cell* to X if $Y = X \sqcup D^n/\!\sim$ where $\sim$ is the equivalence relation given by $x \sim f(x)$ for all $x \in S^{n-1} \subseteq D^n$. We write $Y = X \cup_f e^n$.



2.2 Some terminology:

- The map $f\colon S^{n-1} \to X$ is called the *attaching map* of the cell e^n.
- The map $\bar{f}\colon D^n \to X \sqcup D^n \to X \cup_f e^n$ is called the *characteristic map* of the cell e^n.
- The subspace $e^n = \bar{f}(D^n \smallsetminus S^{n-1}) \subseteq X \cup_f e^n$ is called the *open cell*.
- The subspace $\bar{e}^n = \bar{f}(D^n) \subseteq X \cup_f e^n$ is called the *closed cell*.

2.3 Proposition. *If $f, g\colon S^{n-1} \to X$ are maps such that $f \simeq g$ then $X \cup_f e^n \simeq X \cup_g e^n$.*

2.4 Definition. Let X be topological space and let $A \subseteq X$. The pair (X, A) is a *relative CW complex* if $X = \bigcup_{n=-1}^{\infty} X^{(n)}$ where

1) $X^{(-1)} = A$;
2) for $n \geq 0$ the space $X^{(n)}$ is obtained by attaching n-cells to $X^{(n-1)}$;
3) the topology on X is defined so that a set $U \subseteq X$ is open if and only if $U \cap X^{(n)}$ is open in $X^{(n)}$ for all n.

2.5 Note. If (X, A) is a relative CW complex then the space $X^{(n)}$ is called the *n-skeleton* of X.

2.6 Note. By part 3) of Definition 2.4 if (X, A) is a relative CW complex then a function $f\colon X \to Z$ is continuous if and only if $f|_{X^{(n)}}\colon X^{(n)} \to Z$ is continuous for all $n \geq -1$.

2.7 Note. Assume that (X, A) is a relative CW complex and that we are given a map $g\colon A \to Z$. In such situation, we will often want to construct a map $\bar{g}\colon X \to Z$ such that $\bar{g}|A = g$. Usually, this construction will proceed inductively with respect to the skeleta of X. We will assume that we have already constructed a map $\bar{g}_{n-1}\colon X^{(n-1)} \to Z$ such that $\bar{g}_{n-1}|_A = g$, and we will attempt to extend $\bar{g}_{n-1}$ to $\bar{g}_n\colon X^{(n)} \to Z$. The space $X^{(n)}$ is the quotient space of $X^{(n-1)} \sqcup \bigsqcup_i D^n$ with the equivalence relation defined by the attaching maps of n-cells. Therefore, to define $\bar{g}_n$ it will suffice, for each n-cell e^n with the attaching map $f\colon S^{n-1} \to Z$, to give a map $\varphi\colon D^n \to Z$ such that $\varphi|_{S^{n-1}} = \bar{g}_{n-1}f$.

Once we have maps $\bar{g}_n$ for all n, we can define $\bar{g}\colon X \to Z$ by setting $\bar{g}|_{X^{(n)}} = \bar{g}_n$. The map $\bar{g}$ is continuous by (2.6).

2.8 Definition. A *CW complex* is a space X such that $(X, \varnothing)$ is a relative CW complex.

2.9 Definition. 1) A CW complex X is *finite* if it consists of finitely many cells.

2) A CW complex X is *finite dimensional* if $X = X^{(n)}$ for some n.

3) The *dimension* of a CW complex X is defined by

$$\dim X = \begin{cases} \min\{n \mid X = X^{(n)}\} & \text{if } X \text{ is finite dimensional} \\ \infty & \text{otherwise} \end{cases}$$

2.10 Definition. Let X, Y be relative CW complexes. A map $f\colon X \to Y$ is *cellular* if $f(X^{(n)}) \subseteq Y^{(n)}$ for all $n \geq 0$.

2.11 Cellular Approximation Theorem. *Let X, Y be relative CW complexes. For any map $f\colon X \to Y$ there exists a cellular map $g\colon X \to Y$ such that $f \simeq g$. Moreover, if $A \subseteq X$ is a subcomplex and $f|_A\colon A \to Y$ is a cellular map then g can be selected so that $f|_A = g|_A$ and $f \simeq g$ (rel A).*

2.12 Corollary. *If $n > m$ then every map $f\colon S^m \to S^n$ is homotopic to a constant map.*

Proof. Consider S^n with the structure of a CW complex with one 0-cell and one n-cell. By Theorem 2.11 any map $f\colon S^m \to S^n$ is homotopic to a cellular map. Since the m-skeleton of S^n consists of a single point, such a cellular map is constant. □

2.13 Definition. Let X be a topological space, and let $A \subseteq X$. The pair (X, A) has the *homotopy extension property* if any map

$$h\colon X \times \{0\} \cup A \times [0, 1] \to Y$$

can be extended to a map $\bar{h}\colon X \times [0, 1] \to Y$.

2.14 Theorem. *Any relative CW complex (X, A) has the homotopy extension property.*

2.15 Proposition. *If (X, A) has the homotopy extension property and A is a contractible space, then the quotient map $q\colon X \to X/A$ is a homotopy equivalence.*

2.16 Inductive Homotopy Lemma. *Let (X, A) be a relative CW complex and let $A = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X$ be subcomplexes of X such that $\bigcup_n X_n = X$. Assume that for $n \geq -1$ we have maps $f_n\colon X \to Y$ such that*

1) $f_n|_{X_{n-1}} = f_{n-1}|_{X_{n-1}}$ for all $n \geq 0$
2) $f_n \simeq f_{n-1}$ $(\mathrm{rel}\ X_{n-1})$ for all $n \geq 0$

Let $g\colon X \to Y$ be given by $g(x) = f_n(x)$ if $x \in X_n$. Then g is a continuous function and $f_{-1} \simeq g$ $(\mathrm{rel}\ A)$.

2.17 Example. Take

$$S^n = \left\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\right\}$$

Denote also $S^{-1} = \varnothing$. For each n we have an embedding $j\colon S^n \hookrightarrow S^{n+1}$ given by $j(x_1, \dots, x_{n+1}) = (x_1, \dots, x_{n+1}, 0)$. Define $S^\infty = \bigcup_n S^n$. A set $U \subseteq S^\infty$ is open if for each $n \geq 0$ the set $U \cap S^n$ is open in S^n.

The space S^∞ has a CW complex structure where S^n is the n-skeleton of S^∞.

2.18 Proposition. *S^∞ is a contractible space.*

Proof. Let $x_0 \in S^0 \subseteq S^\infty$. We can assume that S^∞ has a CW complex structure such that x_0 is a 0-cell. By Lemma 2.16 it will suffice to construct functions $f_n\colon S^\infty \to S^\infty$ for $n \geq 0$ such that

1) $f_{-1} = \mathrm{id}_{S^\infty}$
2) $f_n|_{S^n} = x_0$ for all $n \geq 0$
3) $f_n \simeq f_{n-1}$ $(\mathrm{rel}\ S^{n-1})$ for all $n \geq 0$

We will construct functions f_n by induction with respect to n. Assume that we already have a function f_n satisfying the above properties. This, in particular, means that $f_n|_{S^n} = x_0$. We want to get a function f_{n+1} such that $f_{n+1}|_{S^{n+1}} = x_0$ and $f_n \simeq f_{n+1}$ $(\mathrm{rel}\ S^n)$. By Theorem 2.11, the function f_n is homotopic $(\mathrm{rel}\ S^n)$ to a cellular function $g\colon S^\infty \to S^\infty$. The function g restricts to a map $g|_{S^{n+1}}\colon S^{n+1} \to S^{n+2} \subseteq S^\infty$. Using Corollary 2.12 we obtain that there exists a homotopy $h\colon S^{n+1} \times [0, 1] \to S^\infty$ between $g|_{S^{n+1}}$ and the constant map to x_0. We can choose this homotopy so that it is relative to S^n. By Theorem 2.14 we can extend h to a homotopy $\bar{h}\colon S^\infty \times [0, 1] \to S^\infty$. Take $f_{n+1} = \bar{h}_1$.

$\square$