# 20 | Serre Spectral Sequence

The Serre spectral sequence is a special case of the spectral sequence associated to a filtration described in Theorem 19.9.

**20.1 Definition.** Let $p\colon E \to B$ is a Serre fibration where $B$ is a connected CW complex. Let

$$\varnothing = B^{(-1)} \subseteq B^{(0)} \subseteq \dots B$$

be the filtration of $B$ by skeleta. Taking $E^p := p^{-1}(B^{(k)})$ we obtain a filtration of the space $E$:

$$\varnothing = E^{-1} \subseteq E^{0} \subseteq \dots \subseteq E$$

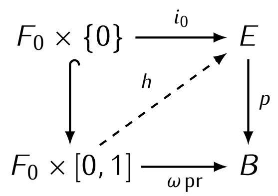
The *Serre spectral sequence* of the fibration $p$ is the spectral sequence associated to this filtration.

By Theorem 19.9 we get that $E^1_{p,q} = H_{p+q}(E^p, E^{p-1})$ and that $E^r_{**}$ converges to $H_*(E)$. The advantage of the Serre spectral sequence is that we can explicitly describe its second page:

**20.2 Theorem.** *Let $E^r_{**}$ be the Serre spectral sequence of a fibration $p\colon E \to B$. Let $F = p^{-1}(b_0)$ for some $b_0 \in B$. If the space $B$ is simply connected then $E^2_{p,q} \cong H_p(B, H_q(F))$.*

While we will skip the proof of this result, it is useful to point out that the assumption that $B$ is simply connected is needed in order to obtain a canonical identification between fibers of $p$ taken over different points. Assume for a moment $p$ is a Hurewicz fibration and that $b_0, b_1 \in B$, Let $F_i = p^{-1}(b_i)$

for $i = 0, 1$. Given a path $\omega\colon [0,1] \to B$ such that $\omega(0) = b_0$ $\omega(1) = b_1$, consider the diagram

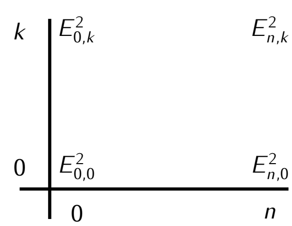



where $\mathrm{pr}\colon F_0 \times [0,1] \to [0,1]$ is the projection map and $i_0\colon F_0 \to E$ is the inclusion. A lift $h$ of $\omega\,\mathrm{pr}$ gives a homotopy in $E$ between the map $i_0$ and a certain map $h_1\colon F_0 \to F_1$. One can show that this map $h_1$ is a homotopy equivalence and that its homotopy class depends only on the homotopy class of the path $\omega$ (relative its endpoints). If the space $B$ is simply connected, all paths joining $b_0$ and $b_1$ are homotopic, so the homotopy class of $h_1$ is uniquely defined. In particular, we obtain canonical isomorphisms of homology groups $h_{1*}\colon H_q(F_0) \xrightarrow{\cong} H_q(F_1)$. If $p$ is a Serre fibration, we can use the same argument, but in order to get the lift $h$ we replace $F_0$ by its CW approximation.

We have seen already one application of the Serre spectral sequence in Theorem 18.7. Here is another one:

**20.3 Proposition.** *Let $S^k \to S^m \xrightarrow{p} S^n$ be a homotopy fibration sequence with $n \geq 1$. Then $k = n - 1$ and $m = 2n - 1$.*

*Proof.* If $n = 1$ then the long exact sequence of homotopy groups shows that we must have $m = 1$ and $k = 0$. Assume then that $n \geq 2$. Consider the Serre spectral sequence of this fibration. Its second page $E^2_{p,q} \cong H_p(S^n, H_q(S^k))$ has only four non-zero terms, all isomorphic to $\mathbb{Z}$:




All differentials originating and terminating at $E^r_{0,0}$ and $E^r_{k,n}$ are trivial, so $E^2_{,0} = E^\infty_{0,0}$ and $E^2_{n,k} = E^\infty_{n,k}$. The page $E^\infty_{**}$ can have non-zero terms $E^\infty_{p,q}$ only if $(p,q) = (0,0)$ or $p + q = m$. It follows that $k + n = m$. The terms $E^2_{0,k}$ and $E^2_{n,0}$ must kill each other, so they must be connected by a differential. This is possible only if $k = n - 1$. Taken together these observations imply that $p$ is a fibration sequence of the form $S^{n-1} \to S^{2n-1} \xrightarrow{p} S^n$. □

Hopf bundles give examples of fibration sequences $S^{n-1} \to S^{2n-1} \to S^n$ for $n = 1, 2, 4, 8$. A theorem of Adams implies that these are the only fibration sequences where all three spaces are spheres.