21 | Serre classes

The motivation for this chapter is to show that the following holds.

21.1 Theorem. *The homotopy groups $\pi_n(S^m)$ are finitely generated for all $n, m \geq 1$.*

This will follow from a more general result that will be stated in terms of Serre classes.

21.2 Definition. A *Serre class* is a non-empty collection $\mathcal{C}$ of abelian groups satisfying the property that if

$$0 \to A \to B \to C \to 0$$

is a short exact sequence of abelian groups then $B \in \mathcal{C}$ if and only if $A, C \in \mathcal{C}$

We will say that a Serre class $\mathcal{C}$ is a *Serre ring* if in addition it satisfies that if $A, B \in \mathcal{C}$ then $A \otimes B \in \mathcal{C}$ and $\mathrm{Tor}(A, B) \in \mathcal{C}$.

We will also say that a Serre class is *acyclic* if for every group $A \in \mathcal{C}$ we have $H_q(K(A, 1)) \in \mathcal{C}$ for all $q > 0$.

21.3 Proposition. *Let $\mathcal{C}$ is a Serre class. The following hold:*

1) $0 \in \mathcal{C}$.
2) *If $A \in \mathcal{C}$ and $A' \cong A$ then $A' \in \mathcal{C}$.*
3) *If $B \subseteq A$ then $A \in \mathcal{C}$ if and only if $B, A/B \in \mathcal{C}$.*
4) *If $A \to B \to C$ is an exact sequence and $A, C \in \mathcal{C}$ then $B \in \mathcal{C}$.*
5) *If $0 = A_{-1} \subseteq A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n$ then $A_n \in \mathcal{C}$ if and only if $A_i/A_{i-1} \in \mathcal{C}$ for all i.*

Proof. Exercise. □

21.4 Proposition. *Let $\mathcal{C}$ is a Serre ring. If X is a path connected space such that $H_q(X) \in \mathcal{C}$ for all $0 < q < p$ then $H_p(X; G) \in \mathcal{C}$ for any group $G \in \mathcal{C}$.*

Proof. By the Universal Coefficient Theorem we have

$$H_p(X; G) \cong (H_p(X) \otimes G) \oplus \mathrm{Tor}(H_{p-1}(X), G)$$

This immediately gives that $H_p(X; G) \in \mathcal{C}$ for $p \geq 2$ For $p = 0$ we gave $H_0(X; G) \cong G \in \mathcal{C}$ while for $p = 1$ we obtain $H_1(X; G) \cong H_1(X) \otimes G \in \mathcal{C}$. □

21.5 Proposition. *All of the following are acyclic Serre rings:*

- $\mathcal{C}_{fin}$ *= the class of all finite abelian groups.*
- $\mathcal{C}_{fg}$ *= the class of all finitely generated abelian groups.*
- $\mathcal{C}_{tor}$ *= the class of all torsion abelian groups.*
- $\mathcal{C}_p$ *= the class of all p-torsion abelian groups for a given prime p.*

21.6 Theorem. *Let $F \to E \xrightarrow{p} B$ be a Serre fibration with a simply connected space B, and let $\mathcal{C}$ be a Serre ring. If for two of the spaces F, E, B the homology groups $H_q(-)$ are in $\mathcal{C}$ for all $q > 0$ then the same holds for the third space.*

Proof. There are three cases to consider.

Case 1: $H_q(F), H_q(B) \in \mathcal{C}$ for all $q > 0$.

Consider the Serre spectral sequence of the fibration p. We have $E^2_{p,q} = H_p(B, H_q(F))$ so by Proposition 21.4 we get that $E^2_{p,q} \in \mathcal{C}$ for all $(p, q) \neq (0, 0)$. Next, since groups $E^3_{p,q}$ are obtained by taking quotients of subgroups of the groups $E^2_{p,q}$, we get that $E^3_{p,q} \in \mathcal{C}$ for all $(p, q) \neq (0, 0)$. Inductively, we obtain that $E^r_{p,q} \in \mathcal{C}$ for all $r \geq 2$ and $(p, q) \neq (0, 0)$, and so also $E^\infty_{p,q} \in \mathcal{C}$ for $(p, q) \neq (0, 0)$. For $q > 0$ the groups $H_q(E)$ admit a finite filtration such filtration quotients are isomorphic to groups $E^\infty_{p,q}$ with $(p, q) \neq 0$. This implies that $H_q(E) \in \mathcal{C}$.

Case 2: $H_q(F), H_q(E) \in \mathcal{C}$ for all $q > 0$.

Since all groups $E^\infty_{p,q}$ are quotients of subgroups of $H_{p+q}(E)$, we have $E^\infty_{p,q} \in \mathcal{C}$ for all $(p, q) \neq (0, 0)$. We will show that $E^2_{p,q} \in \mathcal{C}$ for $(p, q) \neq (0, 0)$ by induction with respect to p. For $p = 0$ this holds since $E^2_{0,q} \cong H_q(F)$. Assume that it also holds for $E^2_{i,q}$ for all $i < p$. It follows that $E^r_{i,q} \in \mathcal{C}$ for all $i < p$ and all $r \geq 2$.

Since all differentials terminating at $E^r_{p,0}$ are trivial, for each r we have an exact sequence

$$E^{r+1}_{p,0} \to E^r_{p,0} \xrightarrow{d^r} E^r_{p-r,r-1}$$

By assumption $E^r_{p-r,r-1} \in \mathcal{C}$, so if $E^{r+1}_{p,0} \in \mathcal{C}$ then the same is true for $E^r_{p,0}$. Since $E^{p+1}_{p,q} = E^\infty_{p,q} \in \mathcal{C}$, arguing inductively over decreasing values of r we obtain that $E^r_{p,0} \in \mathcal{C}$ for all $r \geq 2$. In particular, $H_p(B) = E^2_{p,0} \in \mathcal{C}$. Using Proposition 21.4 we obtain that $E^2_{p,q} = H_p(B, H_q(F)) \in \mathcal{C}$ for all $q \geq 0$.

Case 3: $H_q(B), H_q(E) \in \mathcal{C}$ for all $q > 0$.

This is similar to case 2. □

21.7 Proposition. *If $\mathcal{C}$ is an acyclic Serre ring then for every $A \in \mathcal{C}$ and $n \geq 1$ we have $H_q(K(A,n)) \in \mathcal{C}$.*

Proof. We argue by induction with respect to n. The case $n = 1$ holds by definition of acyclicity of a Serre class. Assume that the statement is true for some $n \geq 1$. For $A \in \mathcal{C}$ consider the homotopy fibration sequence $K(A,n) = \Omega K(A, n+1) \to * \to K(A, n+1)$. Since $H_q(K(A,n)), H_q(*) \in \mathcal{C}$ for all $q > 0$, by Theorem 21.6 we obtain that $H_q(K(A, n+1)) \in \mathcal{C}$. □

21.8 Theorem. *Let $\mathcal{C}$ be an acyclic Serre ring. If X is a simply connected space then the following conditions are equivalent:*

1) *$\pi_n(X) \in \mathcal{C}$ for all $n \geq 1$*
2) *$H_n(X) \in \mathcal{C}$ for all $n \geq 1$*

The proof of Theorem 21.8 will make use of the notion of Postnikov sections:

21.9 Definition. Let X be a path connected space. The n-th Postnikov section of X is a space X_n together with a map $f\colon X \to X_n$ such that

1) $f_*\colon \pi_q(X) \to \pi_q(X_n)$ is an isomorphism for $q \leq n$
2) $\pi_q(X) = 0$ for $q > n$.

The n-th Postnikov section of a space X can be constructed glueing to X cells in dimensions $n+1$ and higher to kill all homotopy groups above $\pi_n(X)$. The map $f\colon X \to X_n$ is then given by the inclusion.

Proof of Theorem 21.8.

1) ⇒ 2) Let X_n denote the n-th Postnikov section of X. By Theorem 16.4 we have $H_q(X) \cong H_q(X_n)$ for all $q < n$, so it will be enough to show that $H_q(X_n) \in \mathcal{C}$ for all $n, q > 0$. We will prove this by induction with respect to n. For $n = 2$ we have $X_2 = K(\pi_2(X), 2)$, so the statement holds by Proposition 21.7. Assume that it also holds for some $n \geq 2$. Notice that we have a fibration sequence

$$K(\pi_{n+1}(X), n+1) \to X_{n+1} \to X_n$$

Using Proposition 21.7 again we get that $H_q(K(\pi_{n+1}(X), n+1)) \in \mathcal{C}$ for $q > 0$, so using Theorem 21.6 we obtain that $H_q(X_{n+1}) \in \mathcal{C}$ for $q > 0$.

2) ⇒ 1) We will show that $\pi_n(X) \in \mathcal{C}$ by induction with respect to n. Since X is simply connected, for $n = 2$ by the Hurewicz Isomorphism Theorem we get $\pi_2(X) \cong H_2(X) \in \mathcal{C}$. Next, assume that $\pi_q(X) \in \mathcal{C}$ for all $q \leq n$ and consider the fibration sequence

$$\operatorname{hofib} f \to X \to X_n$$

where X_n is the n-th Postnikov section of X. Notice that

$$\pi_q(\text{hofib}\, f) = \begin{cases} 0 & if q \leq n \\ \pi_q(X) & if q > n \end{cases}$$

Since $\pi_q(X_n) \in \mathcal{C}$ for all q, thus by part 1) $\Rightarrow$ 2) we get that $H_q(X_n) \in \mathcal{C}$ for all $q > 0$. By assumption $H_q(X) \in \mathcal{C}$ for $q > 0$. Therefore, using Theorem 21.6 we obtain that $H_q(\text{hofib}\, f) \in \mathcal{C}$ for $q > 0$. Since hofib f is n-connected, by the Hurewicz Isomorphism Theorem we get $H_{n+1}(\text{hofib}\, f) \cong \pi_{n+1}(\text{hofib}\, f) \cong \pi_{n+1}(X)$. This gives $\pi_{n+1}(X) \in \mathcal{C}$.

□