

22 | Cohomology via Homotopy

Recall (12.9) that for an abelian group G by $K(G, n)$ we denote the Eilenberg-MacLane space such that $\pi_n(K(G, n)) \cong G$. We will also denote by $K(G, 0)$ the discrete space consisting of elements of the group G . Notice that for every n we have a weak equivalence

$$K(G, n) \xrightarrow{\simeq} \Omega K(G, n+1) \xrightarrow{\simeq} \Omega^2 K(G, n+2)$$

For any pointed CW complex X this induces a bijection of sets of pointed homotopy classes

$$[X, K(G, n)]_* \xrightarrow{\cong} [X, \Omega^2 K(G, n+2)]_*$$

Since $[X, \Omega^2 K(G, n+2)]_*$ has a natural structure of an abelian group (9.8), we obtain in this way an abelian group structure on $[X, K(G, n)]_*$.

The main goal of this chapter is to show that the following holds:

22.1 Theorem. *Let G be an abelian group.*

1) *For any pointed CW complex X and $n \geq 0$ there exists an isomorphism*

$$T_X: [X, K(G, n)]_* \xrightarrow{\cong} \tilde{H}^n(X; G)$$

where $\tilde{H}^n(X; G)$ is the n -th reduced singular cohomology group of X with coefficients in G .

2) *These isomorphisms are natural. That is, if $f: X \rightarrow Y$ is a map of pointed CW complexes then the following diagram commutes:*

$$\begin{array}{ccc} [X, K(G, n)]_* & \xleftarrow{f^*} & [Y, K(G, n)]_* \\ \downarrow T_X \cong & & \cong \downarrow T_Y \\ \tilde{H}^n(X; G) & \xleftarrow{f^*} & \tilde{H}^n(Y; G) \end{array}$$

22.2 Note. Let $\varphi: X \rightarrow K(G, n)$ be a pointed map. By part 2) of Theorem 22.1 we obtain a commutative diagram

$$\begin{array}{ccc} [X, K(G, n)]_* & \xleftarrow{\varphi^*} & [K(G, n), K(G, n)]_* \\ \downarrow T_X \cong & & \downarrow \cong T_{K(G, n)} \\ \tilde{H}^n(X; G) & \xleftarrow{\varphi^*} & \tilde{H}^n(K(G, n); G) \end{array}$$

This gives:

$$T_X([\varphi]) = T_X(\varphi^*([\text{id}_{K(G, n)}])) = \varphi^* T_{K(G, n)}([\text{id}_{K(G, n)}])$$

This implies that for any pointed CW complex X the bijection T_X is determined by the cohomology class $\alpha_n = T_{K(G, n)}([\text{id}_{K(G, n)}]) \in \tilde{H}^n(K(G, n); G)$. This class is called the *fundamental class*.

22.3 Note. An unpointed version of Theorem 22.1 also holds: for any CW complex X there exists a natural isomorphism $T_X: [X, K(G, n)] \xrightarrow{\cong} H^n(X; G)$. This can be derived from Theorem 22.1 as follows. For a CW complex X let X_+ denote the space obtained by adding one 0-cell to X : $X_+ = X \sqcup \{+\}$. We consider $+$ as the basepoint of X_+ . We have bijections $[X, K(G, n)] \cong [X_+, K(G, n)]_*$ and $H^n(X; G) \cong \tilde{H}^n(X_+; G)$. Thus if $[X_+, K(G, n)]_* \cong \tilde{H}^n(X_+; G)$ then $[X, K(G, n)] \cong H^n(X; G)$.

22.4 Example. For any CW complex X we have:

- $[X, S^1] \cong H^1(X, \mathbb{Z})$
- $[X, \mathbb{CP}^\infty] \cong H^2(X, \mathbb{Z})$
- $[X, \mathbb{RP}^\infty] \cong H^2(X, \mathbb{Z}/2)$

The proof of Theorem 22.1 will proceed as follows. First, we will define a general notion of a cohomology theory, which consists of a sequence of functors $\{h^n\}_{n \in \mathbb{Z}}$ from the category of pointed CW complexes to the category of abelian groups satisfying certain axioms. We will show that both assignments $X \mapsto \tilde{H}^n(X; G)$ and $X \mapsto [X, K(G, n)]_*$ satisfy this definition. Then we will prove that if $\{h^n\}$ is any generalized cohomology theory such that $h^n(S^0) \cong \tilde{H}^n(S^0; G)$ for all n , then for every pointed CW complex X and every n there is a natural isomorphism $h^n(X) \rightarrow H^n(X; G)$. Since the cohomology theory defined by Eilenberg-MacLane space satisfies this property, Theorem 22.1 will follow.

22.5 Definition. Let \mathbf{CW}_* denote the category of pointed CW complexes and basepoint preserving maps. A (reduced) cohomology theory consists of:

- A sequence contravariant functors $\{h^n: \mathbf{CW}_* \rightarrow \mathbf{Ab}\}_{n \in \mathbb{Z}}$.
- For every $X \in \mathbf{CW}_*$ and every $n \in \mathbb{Z}$ a natural isomorphism $\Sigma: h^n(X) \rightarrow h^{n+1}(\Sigma X)$. Naturality

means that for any map $f: X \rightarrow Y$ we have a commutative diagram

$$\begin{array}{ccc} h^n(Y) & \xrightarrow{\Sigma} & h^{n+1}(\Sigma Y) \\ f^* \downarrow & \cong & \downarrow \Sigma f^* \\ h^n(X) & \xrightarrow{\Sigma} & h^{n+1}(\Sigma X) \end{array}$$

Moreover, the following axioms are satisfied:

- **(Homotopy axiom)** If $f, g: X \rightarrow Y$ are maps such that $f \simeq g$ then $f^* = g^*: h^n(Y) \rightarrow h^n(X)$ for all n .
- **(Exactness axiom)** For any pair (X, A) where $A \subseteq X$ is a subcomplex, $i: A \hookrightarrow X$ is the inclusion and $q: X \rightarrow X/A$ is the quotient map, the following sequence is exact:

$$h^n(A) \xleftarrow{i^*} h^n(X) \xleftarrow{q^*} h^n(X/A)$$

- **(Wedge axiom)** For any family of pointed CW complexes $\{X_i\}_{i \in I}$ the inclusion maps $X_i \hookrightarrow \bigvee_{i \in I} X_i$ induce isomorphisms $h^n(\bigvee_{i \in I} X_i) \xrightarrow{\cong} \prod_{i \in I} h^n(X_i)$ for all n .

22.6 Some consequences of the axioms.

- $h^n(*) = 0$ for all n .
- For any pair (X, A) where $A \subseteq X$ is a subcomplex, there is a long exact sequence

$$\dots \leftarrow h^n(A) \xleftarrow{i^*} h^n(X) \xleftarrow{q^*} h^n(X/A) \xleftarrow{\delta} h^{n-1}(A) \leftarrow \dots$$

The map $\delta: h^{n-1}(A) \rightarrow h^n(X/A)$ is the composition of the suspension isomorphism $\Sigma: h^{n-1}(A) \rightarrow h^n(\Sigma A)$, the homomorphism induced by the quotient map $C_i \rightarrow C_i/X \cong \Sigma A$, where C_i is the cone of the inclusion $i: A \hookrightarrow X$, and the isomorphism induced by the homotopy equivalence $X/A \xrightarrow{\cong} C_i$.

22.7 Example. Given an abelian group G , consider the reduced singular cohomology functors $X \mapsto \tilde{H}^n(X; G)$. For $n < 0$ set $\tilde{H}^n(X; G) = 0$ for all X . Then the functors $\{\tilde{H}^n(-; G)\}$ define a cohomology theory.

22.8 Example. For an abelian group G , let $h_G^n(X) = [X, K(G, n)]_*$. For $n < 0$ we set $K(G, n) = *$. Then the functors $\{h_G^n\}$ form a cohomology theory. To define the suspension isomorphism

$$\Sigma: h_G^n(X) = [X, K(G, n)]_* \longrightarrow [\Sigma X, K(G, n+1)]_* = h_G^{n+1}(\Sigma X)$$

choose a weak equivalence $\varphi_n: K(G, n) \rightarrow \Omega K(G, n+1)$. This induces an isomorphism $\varphi_n^*: [X, K(G, n)]_* \rightarrow [X, \Omega K(G+1, n)]_*$. Then we compose it with the adjunction isomorphism $[X, \Omega K(G+1, n)]_* \xrightarrow{\cong} [\Sigma X, K(G+1, n)]_*$.

It is obvious that $\{h_G^n\}$ satisfies the homotopy axiom. The exactness axiom is also satisfied by Proposition 10.12. The wedge axiom holds since for any family of well-pointed spaces $\{X_i\}_{i \in I}$ and any pointed space Z , inclusion maps induce a bijection $[\bigvee_{i \in I} X_i, Z]_* \rightarrow \prod_{i \in I} [X_i, Z]_*$.

Notice that the only property of the spaces $K(G, n)$ used in Example 22.8 is that for each n there exist a weak homotopy equivalence $\varphi_n: K(G, n) \rightarrow \Omega K(G, n+1)$. This motivates the following definition.

22.9 Definition. An Ω -spectrum $(K_n, \varphi_n)_{n \in \mathbb{Z}}$ is a sequence of pointed spaces K_n and weak homotopy equivalences $\varphi_n: K_n \xrightarrow{\cong} \Omega K_{n+1}$.

By the same argument as in Example 22.8 we obtain:

22.10 Proposition. Every Ω -spectrum $(K_n, \varphi_n)_{n \in \mathbb{Z}}$ defines a cohomology theory $\{h^n\}_{n \in \mathbb{Z}}$ given by $h^n(X) = [X, K_n]_*$.

22.11 Definition. A cohomology theory $\{h^n\}$ satisfies the *dimension axiom* if $h^n(S^0) = 0$ for $n \neq 0$.

22.12 Theorem. Let $\{h_1^n\}_{n \in \mathbb{Z}}$ and $\{h_2^n\}_{n \in \mathbb{Z}}$ be cohomology theories that satisfy the dimension axiom and such that $h_1^0(S^0) \cong h_2^0(S^0)$. Then for each pointed CW complex there exists natural isomorphism $T_X: h_1^n(X) \xrightarrow{\cong} h_2^n(X)$. Naturality means that each pointed map $f: X \rightarrow Y$ gives a commutative diagram

$$\begin{array}{ccc} h_1^*(Y) & \xrightarrow{f^*} & h_1^*(X) \\ T_Y \downarrow \cong & & \cong \downarrow T_X \\ h_2^*(Y) & \xrightarrow{f^*} & h_2^*(X) \end{array}$$

Proof of Theorem 22.1. For the reduced singular cohomology theory we have

$$\tilde{H}^n(S^0; G) \cong \begin{cases} G & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Also,

$$[S^0, K(G, n)]_* \cong \pi_0(K(G, n)) \cong \begin{cases} G & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Therefore we can apply Theorem 22.12. □

The proof of Theorem 22.12 will require some preparation.

22.13 Lemma. Let $\{h^n\}$ be a cohomology theory satisfying the dimension axiom. Then:

- 1) $h^q(\bigvee_{i \in I} S^n) = 0$ for $q \neq n$.
 2) For any CW complex X the inclusion of the n -th skeleton $j: X^{(n)} \hookrightarrow X$ induces an isomorphism $j^*: h^q(X) \xrightarrow{\cong} h^q(X^{(n)})$ for all $q < n$. Also, $h^q(X^{(n)}) = 0$ for $q > n$.

Proof. 1) This follows from the isomorphisms

$$h^q(\bigvee_{i \in I} S^n) \cong \prod_{i \in I} h^q(S^n) \cong \prod_{i \in I} h^q(\Sigma^n S^0) \cong \prod_{i \in I} h^{q-n}(S^0)$$

2) For finite-dimensional CW complexes this can be proved by induction on skeleta of X , using cofibration sequences $X^{(k-1)} \hookrightarrow X^{(k)} \rightarrow \bigvee S^k$. This can be generalized to the case $\dim X = \infty$ using the infinite telescope construction (see e.g. Hatcher, *Algebraic Topology* pp. 138-139), which gives a cofibration sequence $\bigvee_k X^{(k)} \rightarrow X \rightarrow \bigvee_k \Sigma X^{(k)}$. \square

For abelian groups G, H let $\text{Hom}(G, H)$ denote the set of homomorphisms $G \rightarrow H$. This set has a group structure with addition defined by $(\varphi + \psi)(g) = \varphi(g) + \psi(g)$ for $\varphi, \psi \in \text{Hom}(G, H)$.

22.14 Proposition. Let $\{h^n\}$ be a cohomology theory. For a pointed CW complex X and $n \geq 1$ consider the map

$$\Phi: \pi_n(X) \rightarrow \text{Hom}(h^q(X), h^q(S^n))$$

that sends an element $[\varphi: S^n \rightarrow X] \in \pi_n(X)$ to the induced homomorphism $\varphi^*: h^q(X) \rightarrow h^q(S^n)$. Then the map Φ is a homomorphism of groups.

Proof. The constant map $S^n \rightarrow X$ induces the trivial homomorphism $h^q(X) \rightarrow h^q(S^n)$, so Φ preserves trivial elements. Let $[\varphi], [\psi] \in \pi_n(X)$. The element $[\varphi] \cdot [\psi] \in \pi_n(X)$ is the homotopy class of the map

$$S^n \xrightarrow{p} S^n \vee S^n \xrightarrow{\varphi \vee \psi} S^n$$

where p is the pinch map. We need to show that $p^*(\varphi \vee \psi)^* = \varphi^* + \psi^*: h^q(X) \rightarrow h^q(S^n)$. This follows from commutativity of the following diagram:

$$\begin{array}{ccccc} h^q(X) & \xrightarrow{(\varphi \vee \psi)^*} & h^q(S^n \vee S^n) & \xrightarrow{p^*} & h^q(S^n) \\ & \searrow \varphi^* \times \psi^* & \downarrow \cong & & \nearrow \mu \\ & & h^q(S^n) \times h^q(S^n) & & \end{array}$$

Here the isomorphism $h^q(S^n \vee S^n) \rightarrow h^q(S^n) \times h^q(S^n)$ is induced by the inclusion maps and μ is given by $\mu(x, y) = x + y$. \square

22.15 Corollary. Let $\{h_1^n\}, \{h_2^n\}$ be cohomology theories and let $T: h_1^q(S^n) \rightarrow h_2^q(S^n)$ be an arbitrary homomorphism. Then for any map $f: S^n \rightarrow S^n$ the following diagram commutes:

$$\begin{array}{ccc} h_1^q(S^n) & \xrightarrow{f^*} & h_1^q(S^n) \\ T \downarrow & & \downarrow T \\ h_2^q(S^n) & \xrightarrow{f^*} & h_2^q(S^n) \end{array} \quad (*)$$

Proof. Using Proposition 22.14 we obtain that homotopy classes of maps f for which the diagram $(*)$ commutes form a subgroup of $\pi_n(S^n)$. Since the homotopy class of the identity map $\text{id}_{S^n}: S^n \rightarrow S^n$ belongs to this subgroup, the subgroup contains all elements of $\pi_n(S^n)$. \square

Let $\{h^n\}$ be a cohomology theory and let X be a CW complex. For $n \geq 0$ consider the map

$$\varphi_n: X^{(n+1)}/X^{(n)} \rightarrow \Sigma X^{(n)} \rightarrow \Sigma(X^{(n)}/X^{(n-1)})$$

Let $d^n: h^n(X^{(n)}/X^{(n-1)}) \rightarrow h^{n+1}(X^{(n+1)}/X^{(n)})$ be a homomorphism given by the composition

$$d^n: h^n(X^{(n)}/X^{(n-1)}) \xrightarrow{\Sigma} h^{n+1}(\Sigma(X^{(n)}/X^{(n-1)})) \xrightarrow{\varphi_n^*} h^{n+1}(X^{(n+1)}/X^{(n)})$$

22.16 Proposition. Let $\{h^n\}$ be a cohomology theory. For a CW complex X consider the maps

$$h^{n-1}(X^{(n-1)}/X^{(n-2)}) \xrightarrow{d^{n-1}} h^n(X^{(n)}/X^{(n-1)}) \xrightarrow{d^n} h^{n+1}(X^{(n+1)}/X^{(n)})$$

Then $\text{Im}(d^{n-1}) \subseteq \text{Ker}(d^n)$. Moreover, if $\{h^n\}$ satisfies the dimension axiom then $h^n(X) \cong h^n(X^{(n+1)}) \cong \text{Ker}(d^n)/\text{Im}(d^{n-1})$.

Proof. Exercise. Use Lemma 22.13 and long exact sequences for the pairs $(X^{(n+1)}, X^{(n)})$, $(X^{(n)}, X^{(n-1)})$ and $(X^{(n-1)}, X^{(n-2)})$. \square

Proof of Theorem 22.12. We will construct natural isomorphisms $T_X: h_1^*(X) \rightarrow h_2^*(X)$ in a few steps.

1) We define $T_{S^n}: h_1^*(S^n) \rightarrow h_2^*(S^n)$ by induction with respect to n . By assumption we have isomorphisms $T_{S^0}: h_1^*(S^0) \rightarrow h_2^*(S^0)$. Assume that T_{S^n} is already defined for some n . Choose a homeomorphism $f_{n+1}: S^{n+1} \rightarrow \Sigma S^n$ and define $T_{S^{n+1}}$ so that the following diagram commutes:

$$\begin{array}{ccc} h_1^*(S^n) & \xrightarrow{f_n^* \Sigma} & h_1^*(S^{n+1}) \\ T_{S^n} \downarrow \cong & & \downarrow T_{S^{n+1}} \\ h_2^*(S^n) & \xrightarrow{f_n^* \Sigma} & h_2^*(S^{n+1}) \end{array}$$

2) By the definition of a cohomology theory, for any set J inclusion maps $S^n \rightarrow \bigvee_{j \in J} S^n$ induce isomorphisms $h_i^*(\bigvee_{j \in J} S^n) \xrightarrow{\cong} \prod_{j \in J} h_i^*(S^n)$. Choose isomorphisms $T_{\bigvee_{j \in J} S^n}$ so that the following diagram commutes:

$$\begin{array}{ccc} h_1^*(\bigvee_{j \in J} S^n) & \xrightarrow{\cong} & \prod_{j \in J} h_1^*(S^n) \\ T_{\bigvee_{j \in J} S^n} \downarrow \cong & & \cong \downarrow \prod_{j \in J} T_{S^n} \\ h_2^*(\bigvee_{j \in J} S^n) & \xrightarrow{\cong} & \prod_{j \in J} h_2^*(S^n) \end{array}$$

We claim that isomorphisms $T_{\bigvee_{j \in J} S^n}$ defined above are natural with respect to all maps $f: \bigvee_{j \in J} S^n \rightarrow \bigvee_{k \in K} S^n$. That is, for any such map the following diagram commutes:

$$\begin{array}{ccc} h_1^*(\bigvee_{k \in K} S^n) & \xrightarrow{f^*} & h_1^*(\bigvee_{j \in J} S^n) \\ T_{\bigvee_{k \in K} S^n} \downarrow \cong & & \cong \downarrow T_{\bigvee_{j \in J} S^n} \\ h_2^*(\bigvee_{k \in K} S^n) & \xrightarrow{f^*} & h_2^*(\bigvee_{j \in J} S^n) \end{array}$$

Using the isomorphisms $h_i^*(\bigvee_{j \in J} S^n) \cong \prod_{j \in J} h_i^*(S^n)$ and compactness of spheres, this can be reduced (exercise) to checking that for any pointed map $f: S^n \rightarrow S^n$ we have $f^* T_{S^n} = T_{S^n} f_*$. This, however, follows from Corollary 22.15.

3) There are now two possible ways of obtaining an isomorphism $h_1^*(\Sigma \bigvee_{i \in I} S^n) \rightarrow h_2^*(\Sigma \bigvee_{i \in I} S^n)$. One is to use the suspension isomorphisms $\Sigma: h_k^*(\bigvee_{i \in I} S^n) \rightarrow h_k^{*+1}(\Sigma \bigvee_{i \in I} S^n)$ and the already defined isomorphism $T_{\bigvee_{i \in I} S^n}$. Another is to use the homeomorphism

$$\bigvee_{i \in I} S^{n+1} \xrightarrow{\bigvee f_{n+1}} \bigvee_{i \in I} \Sigma S^n \xrightarrow{\bigvee \Sigma j_i} \Sigma \bigvee_{i \in I} S^n$$

and the isomorphism $T_{\bigvee_{i \in I} S^{n+1}}$. Here $f_{n+1}: S^{n+1} \rightarrow \Sigma S^n$ is a homeomorphism and Σj_i is the suspension of the inclusion map $j_i: S^n \rightarrow \bigvee_{i \in I} S^n$. One can check that both these methods give the same isomorphism $T_{\Sigma \bigvee S^n}: h_1^*(\Sigma \bigvee_{i \in I} S^n) \rightarrow h_2^*(\Sigma \bigvee_{i \in I} S^n)$. Using naturality of isomorphisms $T_{\bigvee S^{n+1}}$ established in 2), we obtain that for any map $f: \bigvee_{j \in J} S^{n+1} \rightarrow \Sigma \bigvee_{k \in K} S^n$ we have a commutative diagram

$$\begin{array}{ccccc} h_1^{*-1}(\bigvee_{k \in K} S^n) & \xrightarrow{\Sigma} & h_1^*(\Sigma \bigvee_{k \in K} S^n) & \xrightarrow{f^*} & h_1^*(\bigvee_{j \in J} S^{n+1}) \\ T_{\bigvee_{k \in K} S^n} \downarrow \cong & & T_{\Sigma \bigvee_{k \in K} S^n} \downarrow \cong & & \cong \downarrow T_{\bigvee_{j \in J} S^{n+1}} \\ h_2^{*-1}(\bigvee_{k \in K} S^n) & \xrightarrow{\Sigma} & h_2^*(\Sigma \bigvee_{k \in K} S^n) & \xrightarrow{f^*} & h_2^*(\bigvee_{j \in J} S^{n+1}) \end{array}$$

4) Let now X be an arbitrary pointed CW complex. By Proposition 22.16 for $k = 1, 2$ we have isomorphisms $h_k^n(X) \cong \text{Ker}(d_k^n) / \text{Im}(d_k^{n-1})$ where $d_k^n: h_k^n(X^{(n)} / X^{(n-1)}) \rightarrow h_k^{n+1}(X^{(n+1)} / X^{(n)})$. If we could

find isomorphisms $T_{X,n}: h_1^n(X^{(n)}/X^{(n-1)}) \rightarrow h_2^n(X^{(n)}/X^{(n-1)})$ such that $d_2^n T_{X,n} = T_{X,n+1} d_1^n$, then they would induce isomorphisms

$$T_X: h_1^n(X) \cong \text{Ker}(d_1^n)/\text{Im}(d_1^{n-1}) \longrightarrow \text{Ker}(d_2^n)/\text{Im}(d_2^{n-1}) \cong h_2^n(X)$$

Such isomorphisms $T_{X,n}$ can be constructed as follows. For each n choose a homeomorphism $f_n: \bigvee_{i \in I_n} S^n \rightarrow X^{(n)}/X^{(n-1)}$. Then define $T_{X,n}$ so that the following diagram commutes:

$$\begin{array}{ccc} h_1^n(X^{(n)}/X^{(n-1)}) & \xrightarrow[\cong]{f_n^*} & h_1^n(\bigvee_{i \in I_n} S^n) \\ T_{X,n} \downarrow & & \downarrow \cong T_{\bigvee_{i \in I_n} S^n} \\ h_2^n(X^{(n)}/X^{(n-1)}) & \xrightarrow[\cong]{f_n^*} & h_2^n(\bigvee_{i \in I_n} S^n) \end{array}$$

Commutativity of isomorphisms $T_{X,n}$ with the maps d_k^n follows from commutativity of the following diagram:

$$\begin{array}{ccccc} & & d_1^n & & \\ & \searrow & \text{---} & \nearrow & \\ h_1^n(X^{(n)}/X^{(n-1)}) & \xrightarrow[\cong]{\Sigma} & h_1^{n+1}(\Sigma(X^{(n)}/X^{(n-1)})) & \xrightarrow{\varphi_n^*} & h_1^{n+1}(X^{(n+1)}/X^{(n)}) \\ & \downarrow f_n^* \cong & \downarrow (\Sigma f_n)^* & & \downarrow f_{n+1}^* \cong \\ h_1^n(\bigvee_{i \in I_n} S^n) & \xrightarrow[\cong]{\Sigma} & h_1^{n+1}(\Sigma \bigvee_{i \in I_n} S^n) & \xrightarrow{((\Sigma f_n)^{-1} \varphi_n f_{n+1})^*} & h_1^{n+1}(\bigvee_{i \in I_{n+1}} S^n) \\ T_{X,n} \downarrow & & \downarrow T_{\Sigma \bigvee_{i \in I_n} S^n} \cong & & \downarrow T_{\bigvee_{i \in I_{n+1}} S^n} T_{X,n+1} \\ h_2^n(\bigvee_{i \in I_n} S^n) & \xrightarrow[\cong]{\Sigma} & h_2^{n+1}(\Sigma \bigvee_{i \in I_n} S^n) & \xrightarrow{((\Sigma f_n)^{-1} \varphi_n f_{n+1})^*} & h_2^{n+1}(\bigvee_{i \in I_{n+1}} S^n) \\ & \uparrow f_n^* \cong & \uparrow (\Sigma f_n)^* & & \uparrow f_{n+1}^* \cong \\ h_2^n(X^{(n)}/X^{(n-1)}) & \xrightarrow[\cong]{\Sigma} & h_2^{n+1}(\Sigma(X^{(n)}/X^{(n-1)})) & \xrightarrow{\varphi_n^*} & h_2^{n+1}(X^{(n+1)}/X^{(n)}) \\ & \nearrow & \text{---} & \searrow & \\ & & d_2^n & & \end{array}$$

The maps φ_n are defined as in Proposition 22.16. The middle squares commute by 3).

To check that the isomorphisms T_X are natural with respect to maps $f: X \rightarrow Y$, notice that we can assume that f is cellular and so it induces homomorphisms $f^*: h_k^n(Y^{(n)}/Y^{(n-1)}) \rightarrow h_k^n(X^{(n)}/X^{(n-1)})$ which commute with the maps d_k^n . Then it remains check that $f^* T_{Y,n} = T_{X,n} f^*$. This can be verified using naturality of the isomorphisms $T_{\bigvee S^n}$ established in 2).

□