5 | Some Computations

5.1 Proposition. *If X is a contractible space then $\pi_n(X) = 0$ for all $n \geq 0$.*

Proof. Since $X \simeq *$ thus $\pi_n(X) \stackrel{\simeq}{=} \pi_n(*) = 0$. □

5.2 Proposition. *If X is a relative CW-complex, $X^{(n)}$ is the n-skeleton of X, and $x_0 \in X^{(n)}$, then the homomorphism $i_*\colon \pi_k(X^{(n)}, x_0) \to \pi_k(X, x_0)$ induced by the inclusion map $i\colon X^{(n)} \hookrightarrow X$ is an isomorphism for $k < n$ and an epimorphism for $k = n$.*

Proof. We can assume that $x_0 \in X^{(0)}$. Consider S^k as a CW complex with a 0-cell $s_0 \in S^k$. By the Cellular Approximation Theorem 2.11 any map $\omega\colon (S^k, s_0) \to (X, x_0)$ is homotopic (relative to the basepoint) to cellular map $\omega'\colon (S^k, s_0) \to (X, x_0)$. If $k \leq n$ then $\omega'(S^k) \subseteq X^{(n)}$, so ω' represents an element of $\pi_k(X^{(n)}, x_0)$ such that $i_*([\omega']) = [\omega]$. This shows that i_* is an epimorphism for $k \leq n$.

Next, take $[\omega_0], [\omega_1] \in \pi_k(X^{(n)}, x_0)$. We can assume that the maps $\omega_0, \omega_1\colon (S^k, s_0) \to (X^{(n)}, x_0)$ are cellular. If $i_*([\omega_0]) = i_*([\omega_1])$ then there is a homotopy $h\colon S^k \times [0, 1] \to X$. Using the Cellular Approximation Theorem 2.11 again, we can assume that this homotopy is a cellular map. Since $\dim S^k \times [0, 1] = k + 1$, we obtain that if $k < n$ then $h(S^k \times [0, 1]) \to X^{(n)}$. Thus h gives a homotopy between ω_0 and ω_1 in $X^{(n)}$. Therefore $[\omega_0] = [\omega_1] \in \pi_k(X^{(n)}, x_0)$. This shows that i_* is a monomorphism for $k < n$. □

5.3 Corollary. *If $k < n$ then $\pi_k(S^n) = 0$*

Proof. A sphere S^n can be given a CW-complex structure with one 0-cell and one n-cell. Then by Proposition 5.2 for $k < n$ we have an epimorphism

$$\pi_k((S^n)^{(k)}) \to \pi_k(S^n)$$

Since $(S^n)^{(k)} = *$, thus $\pi_k((S^n)^{(k)}) = 0$ and so $\pi_k(S^n) = 0$. □

5.4 Definition. A space X is *n-connected* if $\pi_k(X) = 0$ for all $k \leq n$.

Corollary 5.3 can be restated by saying that the sphere S^n is $(n-1)$-connected.

5.5 Proposition. *For any space X and $n \geq 0$ the following conditions are equivalent:*

1) *X is n-connected.*
2) *For any $k \leq n$ and any map $\varphi\colon S^k \to X$ there exists a map $\overline{\varphi}\colon D^{k+1} \to X$ such that $\overline{\varphi}|_{S^k} = \varphi$.*

Proof. Follows from Proposition 3.7. □

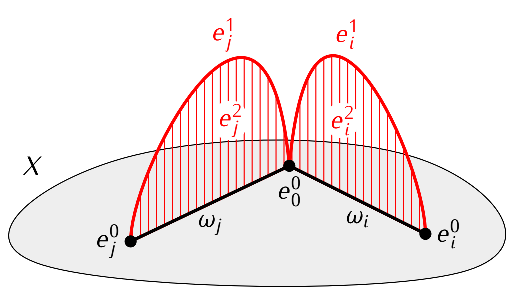
By Proposition 5.2 if X is a CW complex that has only one 0-cell and no k-cells for $k \leq n$ (i.e. $X^{(n)} = *$) then X is n-connected. One can show that the opposite is also true, up to a homotopy equivalence:

5.6 Proposition. *If X is an n-connected CW complex, then there exists a CW complex Y such that such that $X \simeq Y$ and $Y^{(n)} = *$.*

Proof. We will show inductively that for any $k = 0, \dots, n$ there exists a CW complex Y_k such that $X \simeq Y_k$ and $Y_k^{(k)} = *$.

Choose a 0-cell $e^0_0 \in X$. Since $\pi_0(X) = 0$, the space X is path connected. Thus for any 0-cell e^0_i we can select a path $\omega_i\colon [0,1] \to X$ such that $\omega(0) = e_0$ and $\omega(1) = e^0_i$. By the Cellular Approximation Theorem 2.11, we can assume that ω_i is a path in $X^{(1)}$. We construct a new CW complex Y''_0 by attaching cells to X as follows.

1) First, for each 0-cell e^0_i we attach to X a 1-cell e^1_i using the attaching map $\varphi_i\colon S^0 = \{-1, 1\} \to X$ such that $\varphi_i(-1) = e^0_0$ and $\varphi_i(1) = e^0_i$. Let $Y'_0 = X \cup \bigcup_i e^1_i$ be the CW complex obtained in this way.
2) In Y'_0 each 0-cell e^0_i is connected to e^0_0 by two different paths: ω_i, and a path τ_i that traverses the new cell e^1_i. For each i we attach a 2-cell e^2_i using an attaching map $\psi_i\colon S^1 \to Y'_0$ that send the lower half circle to ω_i and the upper half circle to τ_i. Let $Y''_0 = Y'_0 \cup \bigcup_i e^2_i$.



Notice that X is a deformation retract of Y''_0, so the inclusion map $j\colon X \hookrightarrow Y''_0$ is a homotopy equivalence. Also $A = X^{(0)} \cup \bigcup_i e^1_i$ is a contractible subcomplex of Y''_0. By Proposition 2.15, the quotient map

$q\colon Y_0'' \to Y_0''/A$ is a homotopy equivalence. Since Y_0''/A has a CW complex structure with only one 0-cell we can take $Y_0 = Y_0''/A$.

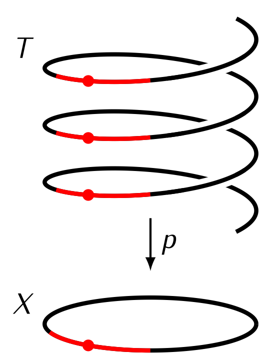
Next, assume that for some $k \leq n$ we have already constructed a CW complex Y_{k-1} such that $X \simeq Y_{k-1}$ and $Y_{k-1}^{(k-1)} = *$. This means that the k-skeleton of Y_{k-1} is given by $Y_{k-1}^{(k)} = \bigvee_i S^k$, with one copy of S^k for each k-cell e_i^k in Y_{k-1}. Let $\varphi_j\colon S^k \hookrightarrow \bigvee_i S^k \subseteq Y_{k-1}$ be the inclusion of the j-th copy of S^k. Since $\pi_k(Y_{k-1}) \cong \pi_k(X) = 0$, each map φ_i extends to a map $\omega_i\colon D^{k+1} \to Y_{k-1}$. We construct a new CW complex Y_k'' by attaching cells to Y_{k-1} as follows.

1) First, for each i we attach a $(k+1)$-cell e_i^{k+1} using $\varphi_i\colon S^k \to Y_{k-1}$ as the attaching map. Let $Y_k' = Y_{k-1} \cup \bigcup_i e_i^{k+1}$ be the CW complex obtained in this way.
2) For each i we have now two maps $D^{k+1} \to Y_k'$: the map ω_i, and the characteristic map τ_i of the cell e_i^{k+1}. Using these maps we attach, for each i, a $(k+2)$-cell e_i^{k+2}, using an attaching map $\psi_i\colon S^{k+1} \to Y_k'$ that sends the lower hemisphere of S^{k+1} to ω_i and the upper hemisphere to τ_i. Let $Y_k'' = Y_k' \cup \bigcup_i e_i^2$.

As before, we observe that Y_{k-1} is a deformation retract of Y_k'', and that $A = Y_{k-1}^{(k)} \cup \bigcup_i e_i^k$ is a contractible subcomplex of Y_k''. Therefore we obtain a $X \simeq Y_{k-1} \simeq Y_k'' \simeq Y_k''/A$. It remains to notice that the space $Y_k = Y_k''/A$ has a CW-complex structure such that $Y_k^{(k)} = *$.

□

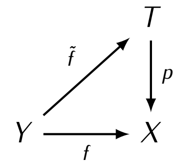
5.7 Homotopy groups and coverings. Recall that covering of a space X is a map $p\colon T \to X$ which is locally homeomorphic to the projection map $\mathrm{pr}_1\colon U \times D \to U$ for some discrete space D.



Recall also, that one of the main properties of coverings is the following fact:

5.8 Theorem (Lifting Criterion). *Let $p\colon T \to X$ be a covering, let $x_0 \in X$ and let $\tilde{x}_0 \in p^{-1}(x_0)$. Assume that Y is a connected and locally path connected space and let $y_0 \in Y$. A map $f\colon (Y, y_0) \to (X, x_0)$*

has a lift $\tilde{f}\colon (Y, y_0) \to (T, \tilde{x}_0)$ *if and only if* $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(T, \tilde{x}_0))$.



Moreover, if a lift $\tilde{f}$ *exists, then it is unique.*

Recall that for any covering $p\colon (T, \tilde{x}_0) \to (X, x_0)$ the induced homomorphism $p_*\colon \pi_1(T, \tilde{x}_0) \to \pi_1(X, x_0)$ is a monomorphism. Using Theorem 5.8 we can generalize this as follows:

5.9 Proposition. *If* $p\colon T \to X$ *is a covering,* $x_0 \in X$ *and* $\tilde{x}_0 \in p^{-1}(x_0)$*, then the induced homomorphism*

$$p_*\colon \pi_n(T, \tilde{x}_0) \to \pi_n(X, x_0)$$

is an isomorphism for all $n > 1$.

Proof. Let $n > 1$ and $\omega\colon (S^n, s_0) \to (X, x_0)$ represents an element of $\pi_n(X, x_0)$. Since $\pi_1(S^n) = 0$, by Theorem 5.8 there exists a map $\widetilde{\omega}\colon (S^n, s_0) \to (T, \tilde{x}_0)$ such that $p\widetilde{\omega} = \omega$. This shows that $p_*\colon \pi_n(T, \tilde{x}_0) \to \pi_n(X, x_0)$ is onto.

Next, assume that $\omega_0, \omega_1\colon (S^n, s_0) \to (T, \tilde{x}_0)$ are maps such that $p_*([\omega_0]) = p_*([\omega_1])$. This means that there exists a basepoint preserving homotopy $h\colon S^n \times [0, 1] \to X$, such that $h_0 = p\omega_0$, $h_1 = p\omega_1$. Since $S^n \times [0, 1] \simeq S^n$ we have $\pi_1(S^n \times [0, 1]) \cong \pi_1(S^n) = 0$. Thus by Theorem 5.8, there exists a homotopy $\widetilde{h}\colon S^n \times [0, 1] \to T$ such that $p\widetilde{h} = h$ and $\widetilde{h}(s_0, 0) = \tilde{x}_0$. Using the uniqueness of lifts, one can check that $\widetilde{h}_0 = \omega_0$ and $\widetilde{h}_1 = \omega_1$, and that the homotopy $\widetilde{h}$ preserves the basepoint (exercise). It follows that $[\omega_0] = [\omega_1]$ in $\pi_1(T, \tilde{x}_0)$. Therefore p_* is a monomorphism.

□

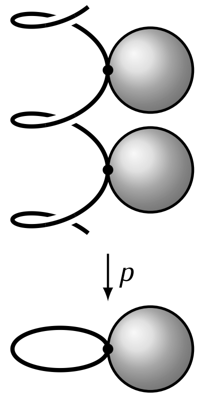
5.10 Example. $\pi_n(S^1) = 0$ for all $n > 1$.

Indeed, universal covering of S^1 is given by a map $p\colon \mathbb{R} \to S^1$. Since $\mathbb{R}$ is a contractible space, by Proposition 5.9 for $n > 1$ we obtain

$$\pi_n(S^1) \cong \pi_n(\mathbb{R}) = 0$$

5.11 Example. If $m > 1$ then $\pi_n(S^1 \vee S^m) \cong \pi_n(\bigvee_{i\in\mathbb{Z}} S^m)$ for all $n > 1$.

To see this, notice that the universal covering of $S^1 \vee S^m$ is the space $\widetilde{X}$ obtained by attaching copies of S^m at all integer points of the real line:



The space $\widetilde{X}$ can be given the structure of a CW complex, such that the real line $\mathbb{R}$ is its subcomplex. Since $\mathbb{R} \simeq \{*\}$, by Theorem 2.14 we have $\widetilde{X} \simeq \widetilde{X}/\mathbb{R} \cong \bigvee_{i\in\mathbb{Z}} S^m$. Therefore for $n > 1$ we obtain

$$\pi_n(S^1 \vee S^m) \cong \pi_n(\widetilde{X}) \cong \pi_n(\bigvee_{i\in\mathbb{Z}} S^m)$$

5.12 Note. Example 5.11 can be used to show that if X, Y are spaces such that $\pi_n(X) \cong \pi_n(Y)$ for all $n \geq 0$, then this does not imply that $X \simeq Y$.

Take, for example, $X = S^1 \vee S^m$ for some $m > 1$, and let $Y = S^1 \vee S^m \vee S^m$. These spaces are not homotopy equivalent, since they have different homology groups: $H_m(X;\mathbb{Z}) \cong \mathbb{Z}$ and $H_m(Y;\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}$.

On the other hand, since both these spaces are path connected, we have $\pi_0(X) \cong \pi_0(Y) \cong \{*\}$. Also, since $\pi_1(S^m) = 0$, thus by van Kampen's theorem we get $\pi_1(X) \cong \pi_1(S^1) \cong \pi_1(Y)$.

The universal covering space $\widetilde{Y}$ of Y is the space obtained by attaching $S^m \vee S^m$ at all integer points of $\mathbb{R}$. Using the same argument as in Example 5.11, we obtain $\widetilde{Y} \simeq \bigvee_{i\in\mathbb{Z}}(S^m \vee S^m) \cong \bigvee_{i\in\mathbb{Z}} S^m$. Therefore for $n \geq 2$ we have

$$\pi_n(X) \cong \pi_n(\bigvee_{i\in\mathbb{Z}} S^m) \cong \pi_n(Y)$$

5.13 Theorem. *For a family $(X_i, \bar{x}_i)_{i\in I}$ be a family of pointed spaces there is an isomorphism*

$$\pi_n\left(\prod_{i\in I} X_i,\ (\bar{x}_i)_{i\in I}\right) \cong \prod_{i\in I} \pi_n(X_i, \bar{x}_i)$$

Proof. For $j \in I$ let $p_j\colon \prod_{i\in I} X_i \to X_j$ denote the projection onto the j-th factor. The induced homomorphisms p_{j*} define a homomorphism:

$$\prod_{i\in I} p_{i*}\colon \pi_n\left(\prod_{i\in I} X_i,\ (\bar{x}_i)_{i\in I}\right) \to \prod_{i\in I} \pi_n(X_i, \bar{x}_i)$$

To obtain a homomorphism going in the opposite direction, let $([\omega_i])_{i\in I}$ be an element of $\prod_{i\in I}\pi_n(X_i,\bar{x}_i)$. Then each ω_i is a map $\omega_i\colon(S^n,s_0)\to(X_i,\bar{x}_i)$. Take the product map

$$\prod_{i\in I}\omega_i\colon(S^n,s_0)\to(\prod_i X_i,\bar{x}_i)$$

One can check that the assignment $([\omega_i])_{i\in I}\mapsto[\prod_{i\in I}\omega_i]$ gives a well-defined homomorphism

$$g\colon\prod_{i\in I}\pi_n(X_i,\bar{x}_i)\to\pi_n\left(\prod_{i\in I}X_i,\ (\bar{x}_i)_{i\in I}\right)$$

and that the compositions $g\circ\prod_{i\in I}p_{i*}$ and $\prod_{i\in I}p_{i*}\circ g$ are identity homomorphisms (exercise). □

5.14 Example. Since $\pi_1(S^1)\cong\mathbb{Z}$ and $\pi_n(S^1)=0$ for $n>1$, thus for any set I we have

$$\pi_n\left(\prod_{i\in I}S^1\right)\cong\begin{cases}\prod_{i\in I}\mathbb{Z} & \text{for } n=1\\ 0 & \text{for } n>1\end{cases}$$