7 | Fibrations

7.1 Definition. A map $p\colon E \to B$ has the *homotopy lifting property* for a space X if for any commutative diagram of the form

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "\bar{f}"] \arrow[d, hook, "i"'] & E \arrow[d, "p"] \\
X \times [0,1] \arrow[r, "h"'] \arrow[ur, dashed, "\bar{h}"] & B
\end{tikzcd}$$

there exists a map $\bar{h}\colon X \times [0,1] \to E$ such that $\bar{h}i = \bar{f}$ and $p\bar{h} = h$.

In the setting of Definition 7.1 we will say that $\bar{h}$ is a lift of h beginning at $\bar{f}$.

7.2 Definition. A map $p\colon E \to B$ is

- a *Hurewicz fibration* if it has the homotopy lifting property for any space X.
- a *Serre fibration* if it has the homotopy lifting property for any CW complex X.

7.3 Note. Every Hurewicz fibration is a Serre fibration.

7.4 Example. For any spaces B, F the projection map $\mathrm{pr}_B\colon B \times F \to B$ is a Hurewicz fibration. Indeed, assume that we have a commutative diagram

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "\bar{f}"] \arrow[d, hook, "i"'] & B \times F \arrow[d, "\mathrm{pr}_B"] \\
X \times [0,1] \arrow[r, "h"'] & B
\end{tikzcd}$$

Let $\mathrm{pr}_F\colon B \times F \to F$ be the projection onto F. We can define $\bar{h}\colon X \times [0,1] \to B \times F$ by

$$\bar{h}(x,t) = (h(x,t),\ \mathrm{pr}_F \bar{f}(x,0))$$

7.5 Example. Every covering map $p\colon E \to B$ is a Hurewicz fibration.

7.6 Definition. Let $A \subseteq X$. A map $p\colon E \to B$ has the *relative homotopy lifting property* for the pair (X,A) if for any commutative diagram of the form

$$\begin{tikzcd}
X \times \{0\} \cup A \times [0,1] \arrow[r, "\bar{f}"] \arrow[d, hook, "i"'] & E \arrow[d, "p"] \\
X \times [0,1] \arrow[r, "h"'] \arrow[ur, dashed, "\bar{h}"] & B
\end{tikzcd}$$

there exists a map $\bar{h}\colon X \times [0,1] \to E$ such that $\bar{h}i = \bar{f}$ and $p\bar{h} = h$.

7.7 Theorem. *Let $p\colon E \to B$ be a map. The following conditions are equivalent:*

1) *p is a Serre fibration;*
2) *p has the homotopy lifting property for D^n for all $n \geq 0$;*
3) *p has the relative homotopy lifting property for (D^n, S^{n-1}) for all $n \geq 0$;*
4) *p has the relative homotopy lifting property for all relative CW-complexes (X,A).*

Proof. 1) $\Rightarrow$ 2) Obvious.

2) $\Rightarrow$ 3) Assume that we have a diagram

$$\begin{tikzcd}
D^n \times \{0\} \cup S^{n-1} \times [0,1] \arrow[r, "\bar{f}"] \arrow[d, hook] & E \arrow[d, "p"] \\
D^n \times [0,1] \arrow[r, "h"'] \arrow[ur, dashed, "\bar{h}"] & B
\end{tikzcd}$$

We want to show that the map $\bar{h}$ exists.

We can construct a homeomorphism $\varphi\colon D^n \times [0,1] \to D^n \times [0,1]$ such that $\varphi(D^n \times \{0\}) = D^n \times \{0\} \cup S^{n-1} \times [0,1]$. This gives a commutative diagram

$$\begin{tikzcd}
D^n \times \{0\} \arrow[r, "\varphi"] \arrow[d, hook] & D^n \times \{0\} \cup S^{n-1} \times [0,1] \arrow[r, "\bar{f}"] \arrow[d, hook] & E \arrow[d, "p"] \\
D^n \times [0,1] \arrow[r, "\varphi"'] \arrow[urr, dashed, "h'"] & D^n \times [0,1] \arrow[r, "h"'] & B
\end{tikzcd}$$

The map $h'\colon D^n \times [0,1] \to E$ exists by 2). Then we can take $\bar{h} = h'\varphi^{-1}$.

3) ⇒ 4) Let (X, A) be a relative complex, and assume that we have a commutative diagram

$$\begin{tikzcd} X \times \{0\} \cup A \times [0,1] \arrow[r, "\bar{f}"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "h"'] \arrow[ur, dashed, "\bar{h}"] & B \end{tikzcd}$$

We want to show that the map $\bar{h}$ exists.

Assume that X is obtained by attaching a single n-dimensional cell e^n to A using an attaching map $\varphi\colon S^{n-1} \to A$, i.e. $X = A \cup_\varphi e^n$. Let $\overline{\varphi}\colon D^n \to X$ be the characteristic map of e^n (2.2). Then the above diagram can be extended as follow:

$$\begin{tikzcd} D^n \times \{0\} \cup S^{n-1} \times [0,1] \arrow[r, "\overline{\varphi} \times \{0\} \cup \varphi \times [0,1]"] \arrow[d, hook] & X \times \{0\} \cup A \times [0,1] \arrow[r, "\bar{f}"] \arrow[d, hook] & E \arrow[d, "p"] \\ D^n \times [0,1] \arrow[r, "\overline{\varphi} \times [0,1]"'] \arrow[urr, dashed, "h'"] & X \times [0,1] \arrow[r, "h"'] & B \end{tikzcd}$$

The map h' exists by 3). Since $X \times [0,1]$ is a quotient space of $A \times [0,1] \sqcup D^n \times [0,1]$, the map

$$\bar{f} \sqcup h'\colon A \times [0,1] \sqcup D^n \times [0,1] \to E$$

defines the desired map $\bar{h}\colon X \times [0,1] \to E$. The general statement can be obtained from here by induction with respect to cell attachments.

4) ⇒ 1) Let X be a CW complex and $A = \varnothing$. Then the relative lifting property for (X, A) is the same as the lifting property for X. □

7.8 Note. Property 3) in Theorem 7.7 can be equivalently stated as follows. Given a cube I^n, let K be a subset of ∂I^n consisting of all but one face of I^n. Then for any commutative diagram

$$\begin{tikzcd} K \arrow[r, "\bar{f}"] \arrow[d, hook] & E \arrow[d, "p"] \\ I^n \arrow[r, "h"'] \arrow[ur, dashed, "\bar{h}"] & B \end{tikzcd}$$

there exists a map $\bar{h}\colon I^n \to E$ such that this diagram commutes.

7.9 Lemma. *Let $p\colon E \to B$ be a Serre fibration. Let $e_0 \in E$ and $b_0 \in B$ be points such that $p(e_0) = b_0$, and let $F = p^{-1}(b_0)$. For any $n \geq 1$ the map $p\colon (E, F, e_0) \to (B, b_0, b_0)$ induces an isomorphism of homotopy groups*

$$p_*\colon \pi_n(E, F, e_0) \longrightarrow \pi_n(B, b_0, b_0) = \pi_n(B, b_0)$$

Proof. To check that $p_*\colon \pi_n(E, F, e_0) \to \pi_n(B, b_0)$ is onto, take a map $\omega\colon (I^n, \partial I^n) \to (B, b_0)$. By the relative homotopy lifting property for $(I^{n-1}, \partial I^{n-1})$, we can find a map $\overline{\omega}\colon I^n \to E$ such that $\overline{\omega}(J^{n-1}) = e_0$ and $p\overline{\omega} = \omega$.

$$\begin{tikzcd}
J^{n-1} = I^{n-1} \times \{1\} \cup \partial I^{n-1} \times [0,1] \arrow[r, "c_{e_0}"] \arrow[d, hook] & E \arrow[d, "p"] \\
I^n = I^{n-1} \times [0,1] \arrow[r, "\omega"'] \arrow[ur, dashed, "\overline{\omega}"] & B
\end{tikzcd}$$

Then $\overline{\omega}$ represents and element of $\pi_n(E, F, e_0)$, and $p_*([\overline{\omega}]) = [\omega]$.

It remains to verify that p_* is 1-1. Assume that $\omega_0, \omega_1\colon (I^n, \partial I^n, J^{n-1}) \to (E, F, e_0)$ be maps such that $p_*([\omega_0]) = p_*([\omega_1])$. Then there exists a homotopy $h\colon I^n \times I \to B$ with $h_0 = p\omega_0$ and $h_1 = p\omega_1$, and such that $h(\partial I^n \times [0,1]) = b_0$. Take the subset $K \subseteq I^n \times I$ given by

$$K = I^n \times \{0,1\} \cup J^{n-1} \times [0,1]$$

Notice that K consists of all faces of the cube $I^{n+1} = I^n \times [0,1]$, except for the face $I^{n-1} \times \{0\} \times [0,1]$. Define $\bar{f}\colon X \to E$ by

$$\bar{f}(x) = \begin{cases} \omega_0(x) & \text{for } x \in I^n \times \{0\} \\ \omega_1(x) & \text{for } x \in I^n \times \{1\} \\ e_0 & \text{for } x \in J^{n-1} \times [0,1] \end{cases}$$

By (7.8) we can find a map $\bar{h}\colon I^{n+1} \to E$ such that $\bar{h}|_K = \bar{f}$ and $p\bar{h} = h$. Such map $\bar{h}$ gives a homotopy between ω_0 and ω_1. Therefore $[\omega_0] = [\omega_1]$ in $\pi_n(E, F, e_0)$. □

7.10 Theorem. *Let $p\colon E \to B$ be a Serre fibration. Let $e_0 \in E$ and $b_0 \in B$ be such that $p(e_0) = b_0$, and let $F = p^{-1}(b_0)$. Let $i\colon F \to E$ be the inclusion map. For any $n \geq 1$ define a homomorphism $\partial\colon \pi_n(B, b_0) \to \pi_{n-1}(F, e_0)$ given by*

$$\partial\colon \pi_n(B, b_0) \xrightarrow{p_*^{-1}} \pi_n(E, F, e_0) \xrightarrow{\partial} \pi_{n-1}(F, e_0)$$

Then the following sequence is exact:

$$\ldots \longrightarrow \pi_{n+1}(B, b_0) \xrightarrow{\partial} \pi_n(F, e_0) \xrightarrow{i_*} \pi_n(E, e_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\partial} \ldots$$
$$\ldots \xrightarrow{p_*} \pi_1(B, b_0) \xrightarrow{\partial} \pi_0(F, e_0) \xrightarrow{i_*} \pi_0(E, e_0) \xrightarrow{p_*} \pi_0(B, b_0)$$

Proof. Exactness in almost all places follows from the exactness of the long exact sequence of the triple (E, F, e_0), and the commutativity of the following diagram:

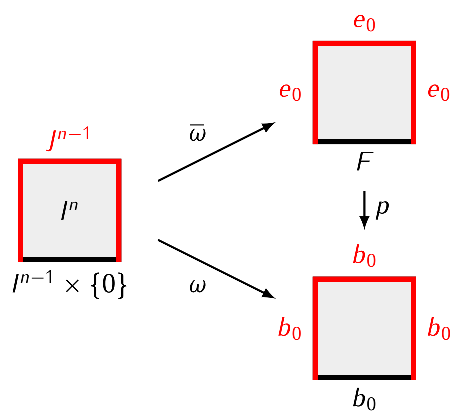
$$\begin{tikzcd}
\dots \arrow[r] & \pi_n(F, e_0) \arrow[r, "i_*"] & \pi_n(E, e_0) \arrow[r, "p_*"] & \pi_n(B, b_0) \arrow[r, "\partial"] & \pi_{n-1}(F, e_0) \arrow[r, "i_*"] & \pi_{n-1}(E, e_0) \arrow[r] & \dots \\
\dots \arrow[r] & \pi_n(F, e_0) \arrow[r, "i_*"'] \arrow[u, "="] & \pi_n(E, e_0) \arrow[r, "j_*"'] \arrow[u, "="] & \pi_n(E, F, e_0) \arrow[r, "\partial"'] \arrow[u, "p_*", "\cong"'] & \pi_{n-1}(F, e_0) \arrow[r, "i_*"'] \arrow[u, "="] & \pi_{n-1}(E, e_0) \arrow[r] \arrow[u, "="] & \dots
\end{tikzcd}$$

Since the long exact sequence of (E, F, x_0) ends at $\pi_0(E, e_0)$, exactness of the sequence

$$\pi_0(F, e_0) \xrightarrow{i_*} \pi_0(E, e_0) \xrightarrow{p_*} \pi_0(B, b_0)$$

needs to be checked separately (exercise). $\square$

7.11 Note. The map $\partial\colon \pi_n(B, b_0) \to \pi_{n-1}(F, e_0)$ can be described directly as follows. Take a map $\omega\colon (I^n, \partial I^n) \to (B, x_0)$. Since $p\colon E \to B$ is a Serre fibration, by (7.8) we can find $\overline{\omega}\colon I^n \to E$ such that $p\overline{\omega} = \omega$, and $\overline{\omega}(J^{n-1}) = e_0$. Then $\partial([\omega]) = [\overline{\omega}|_{I^{n-1}\times\{0\}}]$.



7.12 Example. Consider the product fibration $\mathrm{pr}_B\colon B \times F \to B$. For $b_0 \in B$ we have $\mathrm{pr}_B^{-1}(b_0) = \{b_0\} \times F \cong F$. This for $f_0 \in F$ the exact sequence looks as follows:

$$\dots \to \pi_{n+1}(B, b_0) \xrightarrow{\partial} \pi_n(F, f_0) \xrightarrow{i_*} \pi_n(B \times F, (b_0, f_0)) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\partial} \pi_{n-1}(F, f_0) \xrightarrow{i_*} \dots$$

The projection map $\mathrm{pr}_F\colon B \times F \to F$ induces homomorphisms $\mathrm{pr}_{F*}\colon \pi_n(B \times F, (b_0, f_0)) \to \pi_n(F, f_0)$ such that $\mathrm{pr}_{F*}\, i_* = \mathrm{id}_{\pi_n(F, f_0)}$. This means that $\mathrm{Im}\, \partial = \mathrm{Ker}\, i_* = 0$. Therefore for each $n \geq 1$ we obtain a split short exact sequence

$$0 \longrightarrow \pi_n(F, f_0) \underset{\mathrm{pr}_{F*}}{\overset{i_*}{\rightleftarrows}} \pi_n(B \times F, (b_0, f_0)) \xrightarrow{p_*} \pi_n(B, b_0) \longrightarrow 0$$

This shows that $\pi_n(B \times F, (b_0, f_0)) \cong \pi_n(B, b_0) \times \pi_n(F, f_0)$, which is a special case of the product formula (5.13).

7.13 Example. Let $p\colon E \to B$ be a covering, let $b_0 \in B$ and let $e_0 \in p^{-1}(b_0)$. The space $F = p^{-1}(b_0)$ is discrete, so $\pi_n(F) = 0$ for all $n \geq 1$. Therefore the exact sequence of the fibration becomes

$$\ldots \longrightarrow 0 \longrightarrow \pi_n(E, e_0) \xrightarrow{p_*} \pi_n(B, b_0) \longrightarrow 0 \longrightarrow \ldots$$
$$\ldots \longrightarrow 0 \longrightarrow \pi_1(E, e_0) \xrightarrow{p_*} \pi_1(B, b_0) \xrightarrow{\partial} \pi_0(F, e_0) \xrightarrow{i_*} \pi_0(E, e_0) \xrightarrow{p_*} \pi_0(B, b_0)$$

This shows that $p_*\colon \pi_n(E, e_0) \to \pi_n(B, b_0)$ is an isomorphism for all $n \geq 2$, and it is a monomorphism for $n = 1$. This recovers the statement of Proposition 5.9.

The image of $p_*\colon \pi_1(E, e_0) \to \pi_1(B, b_0)$ coincides with $\mathrm{Ker}(\partial\colon \pi_1(B, b_0) \to \pi_0(F, e_0))$. By the definition of the map ∂, an element $[\omega] \in \pi_1(B, b_0)$ is in $\mathrm{Ker}\,\partial$ if $\omega\colon [0, 1] \to B$ has a lift $\overline{\omega}\colon [0, 1] \to E$ such that $\overline{\omega}(1) = e_0$ and $\omega(1)$ is in the same path connected component of F as e_0. Since F is discrete, it means that $\overline{\omega}(1) = e_0 = \overline{\omega}(0)$. As a consequence, we obtain that $\mathrm{Im}(p_*\colon \pi_1(E, e_0) \to \pi_1(B, b_0)$ consists of elements $[\omega] \in \pi_1(B, b_0)$ such that the lift of ω ending at e_0 is a loop.

7.14 Theorem. *Let $p\colon E \to B$ be map and let $\{U_i\}_{i \in I}$ be an open cover of B. Assume that for each $i \in I$ the map $p_i\colon p^{-1}(U_i) \to U_i$, which is the restriction of p is a Serre fibration. Then p is a Serre fibration.*

7.15 Note. An analogous fact is true for Hurewicz fibrations, under the assumption that B is a paracompact space.

Proof of Theorem 7.14. See e.g. Hatcher *Algebraic Topology*, Proposition 4.48 p. 379. □

7.16 Definition. A map $p\colon E \to B$ is a *fiber bundle* with fiber F if for every point $b \in B$ there exists an open neighborhood $b \in U \subseteq B$ and a homeomorphism $h_U\colon p^{-1}(U) \to U \times F$ such that the following diagram commutes:

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "h_U"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd}$$

Here $\mathrm{pr}_1\colon U \times F \to U$ is the projection map $\mathrm{pr}_1(x, y) = x$.

7.17 Proposition. *Every fiber bundle is a Serre fibration.*

Proof. This follows from Theorem 7.14 and Example 7.4. □

7.18 Example. Every covering space $p\colon E \to B$ is a fiber bundle whose fiber is a discrete space.

7.19 Example. Mobius band

7.20 Example. Klein bottle

7.21 Example. Consider S^{2n+1} as a subspace of the complex space $\mathbb{C}^n$:

$$S^{2n+1} = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \mid \sum_{i=0}^{n} \|z_i\|^2 = 1\}$$

In particular, $S^1 = \{z \in \mathbb{C} \mid \|z\| = 1\}$. The *n-dimensional complex projective space* is the quotient space

$$\mathbb{CP}^n = S^{2n+1}/\sim$$

where $(z_1, \dots, z_n) \sim \lambda(z_0, \dots, z_n)$ for all $\lambda \in S^1$. We will denote by $[z_0, \dots, z_n] \in \mathbb{CP}^n$ the equivalence class of $(z_0, \dots, z_n)$. Let $p\colon S^{2n+1} \to \mathbb{CP}^n$ be the quotient map $p(z_0, \dots, z_n) = [z_0, \dots, z_n]$.

We will show that $p\colon S^{2n+1} \to \mathbb{CP}^n$ is a fiber bundle with fiber S^1. Let $b = [z_0, \dots, z_n] \in \mathbb{CP}^n$. Choose $0 \leq i \leq n$ such that $z_i \neq 0$, and take $U_i = \{[w_0, \dots, w_n] \in \mathbb{CP}^n \mid w_i \neq 0\}$. This set is an open neighborhood of b in $\mathbb{CP}^n$. We have

$$p^{-1}(U_i) = \{(w_0, \dots, w_n) \in S^{2n+1} \mid w_i \neq 0\}$$

Define a map $h_i\colon p^{-1}(U_i) \to U_i \times S^1$ by $h_i(w_0, \dots, w_n) = ([w_0, \dots, w_n], w_i/\|w_i\|)$. This is a homeomorphism, with the inverse given by

$$h_i^{-1}([v_0, \dots, v_n], \lambda) = \frac{\|v_i\|}{v_i} \cdot \lambda \cdot (v_0, \dots, v_n)$$

Let $n \geq 1$. The long exact sequence of the bundle $p\colon S^{2n+1} \to \mathbb{CP}^n$ has the form

$$\dots \longrightarrow \pi_m(S^1) \xrightarrow{i_*} \pi_m(S^{2n+1}) \xrightarrow{p_*} \pi_m(\mathbb{CP}^n) \xrightarrow{\partial} \pi_{m-1}(S^1) \longrightarrow \dots$$
$$\dots \longrightarrow \pi_2(S^{2n+1}) \xrightarrow{p_*} \pi_2(\mathbb{CP}^n) \xrightarrow{\partial} \pi_1(S^1) \xrightarrow{i_*} \pi_1(S^{2n+1}) \xrightarrow{p_*} \pi_1(\mathbb{CP}^n, b_0) \xrightarrow{\partial} \pi_0(S^1) = 0$$

Since $\pi_m(S^1) = 0$ for $m > 1$, we obtain that $\pi_m(\mathbb{CP}^n) \cong \pi_m(S^{2n+1})$ for $m \geq 3$. Also, since $\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_m(S^{2n+1}) = 0$ for $m < 2n+1$, thus $\pi_2(\mathbb{CP}^n) \cong \pi_1(S^1) \cong \mathbb{Z}$ and $\pi_1(\mathbb{CP}^n) = 0$.

7.22 Example. As a special case of Example 7.21, take $n = 1$. In this case, we have a homeomorphism $\mathbb{CP}^1 \cong S^2$. To see this, define a map $h\colon \mathbb{CP}^1 \smallsetminus \{[1,0]\} \to \mathbb{C}$ by $h([z_0, z_1]) = \frac{z_0}{z_1}$. This is a homeomorphism with the inverse given by $h^{-1}(z) = \frac{1}{1+\|z\|} \cdot [z, 1]$. Since S^2 is homeomorphic to the one-point compactification of $\mathbb{C}$, i.e. $S^2 \cong \mathbb{C} \cup \{\infty\}$, the map h can be extended to a homeomomrphism $h\colon \mathbb{CP}^1 \to S^2$ by setting $h([1,0]) = \infty$.

Under the identification $\mathbb{CP}^1 \cong S^2$ the bundle $S^1 \to S^3 \xrightarrow{p} \mathbb{CP}^1$ becomes $S^1 \to S^3 \xrightarrow{p} S^2$. This bundle is called the *Hopf bundle* (or the *Hopf fibration*).

Using the long exact sequence of the Hopf fibration we obtain:

7.23 Theorem. $\pi_2(S^2) \cong \mathbb{Z}$.